Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

November 21, 2025

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the $A = QR$ factorization where A is the 5×5 matrix, Q is the 5×2 matrix, and R is the 2×5 matrix given by

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 & 4 \\ 1 & -1 & 1 & 2 & 4 \\ 3 & 5 & 1 & 4 & 6 \\ 1 & 7 & -1 & 0 & -2 \\ 2 & -2 & 2 & 4 & 8 \end{bmatrix} \qquad Q = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 3 & -1 \\ 1 & -3 \\ 2 & 2 \end{bmatrix} \qquad R = \begin{bmatrix} 4 & 4 & 2 & 6 & 10 \\ 0 & -8 & 2 & 2 & 6 \end{bmatrix}$$

Do not ignore the factor of $1/4$ **used to define** Q **(for instance, the** $(4,2)$ **entry of** Q **is** $-3/4$**)!**

(4 pts) (*a*) $\operatorname{rank}(A) =$ ___2___ and $\det(Q^\intercal Q) =$ ___1___

(4 pts) (*b*) Suppose that $\boldsymbol{b}$ is any vector in the *column space* of A. It is guaranteed that $\boldsymbol{b}$ satisfies exactly one of the following equations. Select this equation.

○ $A\boldsymbol{b} = \boldsymbol{b}$ √ $QQ^\intercal \boldsymbol{b} = \boldsymbol{b}$ ○ $R\boldsymbol{b} = A\boldsymbol{b}$ ○ $Q^\intercal \boldsymbol{b} = R\boldsymbol{b}$ ○ $Q^\intercal \boldsymbol{b} = \boldsymbol{O}$

(4 pts) (*c*) Suppose $\widehat{\boldsymbol{x}}$ is any solution to the least squares problem associated to $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} \in \mathbb{R}^5$ is any vector. It is guaranteed that $\widehat{\boldsymbol{x}}$ solves *all but one* of the following equations. Select the equation that $\widehat{\boldsymbol{x}}$ is ***not*** *guaranteed to solve*.

○ $A\widehat{\boldsymbol{x}} = QQ^\intercal \boldsymbol{b}$ ○ $A^\intercal A\widehat{\boldsymbol{x}} = A^\intercal \boldsymbol{b}$ ○ $R\widehat{\boldsymbol{x}} = Q^\intercal \boldsymbol{b}$ ○ $R^\intercal R\widehat{\boldsymbol{x}} = A^\intercal \boldsymbol{b}$ √ $R^\intercal R\widehat{\boldsymbol{x}} = R^\intercal \boldsymbol{b}$

(6 pts) (*d*) The coefficient of t^4 in $\chi_A(t)$ is ___-9___ and the constant coefficient in $\chi_A(t)$ is ___0___.

(10 pts) (*e*) It is known that $U = I_5 - c \cdot QQ^\intercal$ is real-symmetric for any real number c. However, the matrix U has orthonormal columns for only one *nonzero* real number of c. Find this value of c. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

Solution. We are given that $U = I_5 - c \cdot QQ^\intercal$ is real-symmetric, which means $U^\intercal = U$ and that Q is part of the given $A = QR$ factorization, which means that $Q^\intercal Q = I_2$. We wish to find the nonzero value of c that forces U to have orthonormal columns, which is governed by whether or not $U^\intercal U = I_5$. The relevant calculation here is

$$\begin{aligned} U^\intercal U &= UU \\ &= (I_5 - c \cdot QQ^\intercal)(I_5 - c \cdot QQ^\intercal) \\ &= I_5(I_5 - c \cdot QQ^\intercal) - c \cdot QQ^\intercal(I_5 - c \cdot QQ^\intercal) \\ &= I_5 - c \cdot QQ^\intercal - c \cdot QQ^\intercal + c^2 \cdot Q \underbrace{Q^\intercal Q}_{=I_2} Q^\intercal \\ &= I_5 - 2\,c \cdot QQ^\intercal + c^2 \cdot QQ^\intercal \\ &= I_5 + (c^2 - 2\,c) \cdot QQ^\intercal \\ &= I_5 + c \cdot (c - 2) \cdot QQ^\intercal \end{aligned}$$

From here we see that the only values of c resulting in $U^\intercal U = I_5$ are $c = 0$ and $c = 2$. We are asked for the *nonzero* value, which is $c = 2$.

$c =$ ___2___

Problem 2. The equation below depicts a diagonalization $A = XDX^{-1}$ of a 4×4 complex matrix A.

$$\begin{bmatrix} & & & \\ & A & & \\ & & & \end{bmatrix} = \overset{X}{\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -2 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & 2i+1 & 2i+3 \end{bmatrix}} \overset{D}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i+2 & 0 \\ 0 & 0 & 0 & -2i+2 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} -6i-11 & -6i-12 & 4 & -3 \\ -2i-5 & -2i-5 & 2 & -1 \\ -2i-4 & -2i-4 & 1 & -1 \\ 2i+3 & 2i+3 & -1 & 1 \end{bmatrix}}$$

Throughout this problem, let $\boldsymbol{v}_3 = \begin{bmatrix} -1 & 0 & -2 & 2i+1 \end{bmatrix}^\intercal$ (the third column of X) and let $\boldsymbol{v}_4 = \begin{bmatrix} 1 & -2 & -1 & 2i+3 \end{bmatrix}^\intercal$ (the fourth column of X).

(6 pts) (*a*) $\|\boldsymbol{v}_3\| =$ $\underline{\sqrt{10}}$ and the coefficient of t^3 in $\chi_A(t)$ is $\underline{\quad i-6 \quad}$

(9 pts) (*b*) Some, but not necessarily all, of the following descriptors accurately describe A. Select these descriptors (1.5pts each).

√ diagonalizable ○ real-symmetric ○ Hermitian √ nonsingular

○ indefinite √ has at least one nonreal entry

(8 pts) (*c*) Calculate $\langle \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle$. Clearly explain your reasoning to receive credit. Record your value of $\langle \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle$ in the blank below for clarity.

Solution.

$$\begin{aligned} \langle \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle &= \overline{(-1)}(1) + \overline{(0)}(-2) + \overline{(-2)}(-1) + \overline{(2i+1)}(2i+3) \\ &= -1 + 2 + (-2\,i+1)(2\,i+3) \\ &= 1 + (-4\,i^2 - 6\,i + 2\,i + 3) \\ &= 1 + 4 - 4\,i + 3 \\ &= 8 - 4\,i \end{aligned}$$

$\langle \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle =$ $\underline{\quad -4i+8 \quad}$

(10 pts) (*d*) We can infer from the given diagonalization that $\lambda = 1$ is an eigenvalue of A. Find an *orthonormal basis* of $\mathcal{E}_A(1)$. Clearly explain your reasoning to receive credit. List your basis vectors in the box at the bottom of this page for clarity.

Solution. The eigenvalue $\lambda = 1$ is in the first two diagonal entries of D, so the first two columns $\boldsymbol{v}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^\intercal$ and $\boldsymbol{v}_2 = \begin{bmatrix} -1 & 1 & 1 & 1 \end{bmatrix}^\intercal$ of X form a basis of $\mathcal{E}_A(1)$. To convert this to an *orthonormal* basis, we apply the Gram-Schmidt algorithm.

$$\begin{aligned} \boldsymbol{w}_1 &= \boldsymbol{v}_1 \\ &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^\intercal \\ \boldsymbol{w}_2 &= \boldsymbol{v}_2 - \operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2) \\ &= \boldsymbol{v}_2 - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_2 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 \\ &= \boldsymbol{v}_2 - \frac{-2}{2} \boldsymbol{w}_1 \\ &= \begin{bmatrix} -1 & 1 & 1 & 1 \end{bmatrix}^\intercal + \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^\intercal \\ &= \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\intercal \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_1 &= \frac{1}{\|\boldsymbol{w}_1\|} \boldsymbol{w}_1 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^\intercal \\ \boldsymbol{q}_2 &= \frac{1}{\|\boldsymbol{w}_2\|} \boldsymbol{w}_2 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^\intercal \end{aligned}$$

Our orthonormal basis of $\mathcal{E}_A(1)$ is $\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ for the $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ above.

(18 pts) **Problem 3.** Consider the matrix A and the vector $\boldsymbol{b}$ given by

$$A = \begin{bmatrix} -5 & 9 \\ 1 & -5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 3\,e^8 + 9\,e^2 \\ -e^8 + 3\,e^2 \end{bmatrix}$$

Calculate the matrix-vector product $\exp(A)\boldsymbol{b}$. Clearly explain your reasoning to receive credit. Your answer should simplify to a vector of integers. Record your answer in the blank at the bottom of this page for clarity.

Solution. This is a matrix exponentials problem, which means we should start by diagonalizing A. The characteristic polynomial is

$$\chi_A(t) = t^2 - \operatorname{trace}(A)\,t + \det(A) = t^2 + 10t + 16 = (t+2)\cdot(t+8)$$

This demonstrates that $\text{E-Vals}(A) = \{-2, -8\}$. The *eigenspaces* are

$$\mathcal{E}_A(-2) = \operatorname{Null} \overset{-2\cdot I_2 - A}{\begin{bmatrix} 3 & -9 \\ -1 & 3 \end{bmatrix}} = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{E}_A(-8) = \operatorname{Null} \overset{-8\cdot I_2 - A}{\begin{bmatrix} -3 & -9 \\ -1 & -3 \end{bmatrix}} = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$$

Our diagonalization is then

$$\overset{A}{\begin{bmatrix} -5 & 9 \\ 1 & -5 \end{bmatrix}} = \overset{X}{\begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix}} \overset{D}{\begin{bmatrix} -2 & 0 \\ 0 & -8 \end{bmatrix}} \overset{X^{-1}}{\frac{1}{-6}\begin{bmatrix} -1 & -3 \\ -1 & 3 \end{bmatrix}}$$

We know now from class that $\exp(A) = X\exp(D)X^{-1}$, so our desired matrix-vector product is

$$\begin{aligned} \exp(A)\boldsymbol{b} &= \begin{bmatrix} X \end{bmatrix} \begin{bmatrix} \exp(D) \end{bmatrix} \frac{1}{-6} \begin{bmatrix} -1 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3\,e^8 + 9\,e^2 \\ -e^8 + 3\,e^2 \end{bmatrix} \\ &= \begin{bmatrix} X \end{bmatrix} \begin{bmatrix} \exp(D) \end{bmatrix} \frac{1}{-6} \begin{bmatrix} -18\,e^2 \\ -6\,e^8 \end{bmatrix} \\ &= \begin{bmatrix} X \end{bmatrix} \begin{bmatrix} e^{(-2)} & 0 \\ 0 & e^{(-8)} \end{bmatrix} \begin{bmatrix} 3\,e^2 \\ e^8 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 12 \\ 2 \end{bmatrix} \end{aligned}$$

$$\exp(A)\boldsymbol{b} = \begin{bmatrix} 12 \\ 2 \end{bmatrix}$$

Problem 4. The data below depicts a matrix A (whose three columns are labeled as $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$) along with the quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x}\rangle$ where $S = A^\intercal A$ (which recall means that the (i,j) entry of S is $\langle \boldsymbol{a}_i, \boldsymbol{a}_j\rangle$) and $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\intercal$.

$$A = \begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix} \qquad q(\boldsymbol{x}) = 3\,x_1^2 - 2\,x_1x_2 + 5\,x_2^2 + 2\,x_1x_3 - 2\,x_2x_3 + 3\,x_3^2$$

It is known that the technique of "completing the square" allows one to rewrite this quadratic form as

$$q(\boldsymbol{x}) = \lambda_1 \cdot \left(\frac{x_1 - 2\,x_2 + x_3}{\sqrt{6}}\right)^2 + 3 \cdot \left(\frac{x_1 + x_2 + x_3}{\sqrt{3}}\right)^2 + 2 \cdot y_3^2$$

Note that the symbols λ_1 and y_3 in this presentation of $q(\boldsymbol{x})$ are currently unknown.

(6 pts) (*a*) $\|\boldsymbol{a}_1\|^2 = \|\boldsymbol{a}_3\|^2 =$ __3__ and $\langle \boldsymbol{a}_1, \boldsymbol{a}_2\rangle =$ __-1__

(9 pts) (*b*) Find the value of λ_1. Clearly explain your reasoning to receive credit. Fill your answer in the blank below for clarity.

Solution. The quickest way of doing this is to use the trace formula for eigenvalues. The diagonal entries of S are the coefficients of the square terms in $q(\boldsymbol{x})$, which means $\operatorname{trace}(S) = 3 + 5 + 3 = 11$. The given presentation of $q(\boldsymbol{x})$ after "completing the square" tells us that $\text{E-Vals}(S) = \{\lambda_1, 3, 2\}$. The trace formula for eigenvalues then implies

$$11 = \operatorname{trace}(S) = \lambda_1 + 3 + 2$$

so $\lambda_1 = 6$.

Alternatively, the given presentation of $q(\boldsymbol{x})$ after "completing the square" tells us that $y_1 = \frac{x_1 - 2\,x_2 + x_3}{\sqrt{6}}$, so we expect $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^\intercal$ to be an eigenvector of S corresponding to λ_1. The relevant calculation here is

$$\overset{S}{\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}} \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 6 \\ -12 \\ 6 \end{bmatrix} = \frac{6}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Again, we conclude that $\lambda_1 = 6$.

$\lambda_1 =$ __6__

(6 pts) (*c*) There are only two valid formulas for y_3 in terms of x_1, x_2, and x_3. Find one of these formulas. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

Solution. The change of variables in "completing the square" is $\boldsymbol{y} = U^\intercal \boldsymbol{x}$ where $S = UDU^T$ is a spectral factorization of S. The given presentation of $q(\boldsymbol{x})$ after "completing the square" tells us that $\text{E-Vals}(S) = \{\lambda_1, \lambda_2 = 3, \lambda_3 = 2\}$. The third coordinate of $\boldsymbol{y}$ is then the inner product of any unit basis vector of $\mathcal{E}_S(2)$ with $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\intercal$. To find a unit basis vector of $\mathcal{E}_S(2)$, note that

$$S = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \qquad \mathcal{E}_S(2) = \operatorname{Null} \overset{2\cdot I_3 - S}{\begin{bmatrix} -1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{bmatrix}} = \operatorname{Span}\left\{ \pm \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Here, our basis of $\mathcal{E}_S(2)$ is inferred from the fact that the first and third columns in $2 \cdot I_3 - S$ are equal. This tells us that the two valid equations for y_3 are $y_3 = \pm\left(\frac{x_1 - x_3}{\sqrt{2}}\right)$.

$y_3 =$ __$\pm\left(\frac{x_1 - x_3}{\sqrt{2}}\right)$__