Duke University

Math 218D-2

Matrices and Vectors

Exam III

Name: ______________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

November 21, 2025

- There are 100 points and 4 problems on this 50-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the $A = QR$ factorization where A is the 5×5 matrix, Q is the 5×2 matrix, and R is the 2×5 matrix given by

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 & 4 \\ 1 & -1 & 1 & 2 & 4 \\ 3 & 5 & 1 & 4 & 6 \\ 1 & 7 & -1 & 0 & -2 \\ 2 & -2 & 2 & 4 & 8 \end{bmatrix} \qquad Q = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 3 & -1 \\ 1 & -3 \\ 2 & 2 \end{bmatrix} \qquad R = \begin{bmatrix} 4 & 4 & 2 & 6 & 10 \\ 0 & -8 & 2 & 2 & 6 \end{bmatrix}$$

Do not ignore the factor of $1/4$ used to define Q (for instance, the $(4, 2)$ entry of Q is $-3/4$)!

(4 pts) (a) $\operatorname{rank}(A) =$ ________ and $\det(Q^\intercal Q) =$ ________

(4 pts) (b) Suppose that $\boldsymbol{b}$ is any vector in the *column space* of A. It is guaranteed that $\boldsymbol{b}$ satisfies exactly one of the following equations. Select this equation.

○ $A\boldsymbol{b} = \boldsymbol{b}$ ○ $QQ^\intercal\boldsymbol{b} = \boldsymbol{b}$ ○ $R\boldsymbol{b} = A\boldsymbol{b}$ ○ $Q^\intercal\boldsymbol{b} = R\boldsymbol{b}$ ○ $Q^\intercal\boldsymbol{b} = \boldsymbol{O}$

(4 pts) (c) Suppose $\widehat{\boldsymbol{x}}$ is any solution to the least squares problem associated to $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} \in \mathbb{R}^5$ is any vector. It is guaranteed that $\widehat{\boldsymbol{x}}$ solves *all but one* of the following equations. Select the equation that $\widehat{\boldsymbol{x}}$ is ***not*** *guaranteed to solve.*

○ $A\widehat{\boldsymbol{x}} = QQ^\intercal\boldsymbol{b}$ ○ $A^\intercal A\widehat{\boldsymbol{x}} = A^\intercal\boldsymbol{b}$ ○ $R\widehat{\boldsymbol{x}} = Q^\intercal\boldsymbol{b}$ ○ $R^\intercal R\widehat{\boldsymbol{x}} = A^\intercal\boldsymbol{b}$ ○ $R^\intercal R\widehat{\boldsymbol{x}} = R^\intercal\boldsymbol{b}$

(6 pts) (d) The coefficient of t^4 in $\chi_A(t)$ is ________ and the constant coefficient in $\chi_A(t)$ is ________.

(10 pts) (e) It is known that $U = I_5 - c \cdot QQ^\intercal$ is real-symmetric for any real number c. However, the matrix U has orthonormal columns for only one *nonzero* real number of c. Find this value of c. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

$c =$ ________

Problem 2. The equation below depicts a diagonalization $A = XDX^{-1}$ of a 4×4 complex matrix A.

$$\begin{bmatrix} & & & \\ & A & & \\ & & & \\ & & & \end{bmatrix} = \overset{X}{\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -2 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & 2i+1 & 2i+3 \end{bmatrix}} \overset{D}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i+2 & 0 \\ 0 & 0 & 0 & -2i+2 \end{bmatrix}} \overset{X^{-1}}{\begin{bmatrix} -6i-11 & -6i-12 & 4 & -3 \\ -2i-5 & -2i-5 & 2 & -1 \\ -2i-4 & -2i-4 & 1 & -1 \\ 2i+3 & 2i+3 & -1 & 1 \end{bmatrix}}$$

Throughout this problem, let $\boldsymbol{v}_3 = \begin{bmatrix} -1 & 0 & -2 & 2i+1 \end{bmatrix}^\intercal$ (the third column of X) and let $\boldsymbol{v}_4 = \begin{bmatrix} 1 & -2 & -1 & 2i+3 \end{bmatrix}^\intercal$ (the fourth column of X).

(6 pts) (a) $\|\boldsymbol{v}_3\| =$ _______ and the coefficient of t^3 in $\chi_A(t)$ is _______________

(9 pts) (b) Some, but not necessarily all, of the following descriptors accurately describe A. Select these descriptors (1.5pts each).

○ diagonalizable ○ real-symmetric ○ Hermitian ○ nonsingular

○ indefinite ○ has at least one nonreal entry

(8 pts) (c) Calculate $\langle \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle$. Clearly explain your reasoning to receive credit. Record your value of $\langle \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle$ in the blank below for clarity.

$\langle \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle =$ _______________

(10 pts) (d) We can infer from the given diagonalization that $\lambda = 1$ is an eigenvalue of A. Find an *orthonormal basis* of $\mathcal{E}_A(1)$. Clearly explain your reasoning to receive credit. List your basis vectors in the box at the bottom of this page for clarity.

Orthonormal Basis of $\mathcal{E}_A(1)$

(18 pts) **Problem 3.** Consider the matrix A and the vector $\boldsymbol{b}$ given by

$$A = \begin{bmatrix} -5 & 9 \\ 1 & -5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 3\,e^8 + 9\,e^2 \\ -e^8 + 3\,e^2 \end{bmatrix}$$

Calculate the matrix-vector product $\exp(A)\boldsymbol{b}$. Clearly explain your reasoning to receive credit. Your answer should simplify to a vector of integers. Record your answer in the blank at the bottom of this page for clarity.

$$\exp(A)\boldsymbol{b} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Problem 4. The data below depicts a matrix A (whose three columns are labeled as $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$) along with the quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, S\boldsymbol{x} \rangle$ where $S = A^\intercal A$ (which recall means that the (i, j) entry of S is $\langle \boldsymbol{a}_i, \boldsymbol{a}_j \rangle$) and $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\intercal$.

$$A = \begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix} \qquad q(\boldsymbol{x}) = 3\,x_1^2 - 2\,x_1x_2 + 5\,x_2^2 + 2\,x_1x_3 - 2\,x_2x_3 + 3\,x_3^2$$

It is known that the technique of "completing the square" allows one to rewrite this quadratic form as

$$q(\boldsymbol{x}) = \lambda_1 \cdot \left(\frac{x_1 - 2\,x_2 + x_3}{\sqrt{6}} \right)^2 + 3 \cdot \left(\frac{x_1 + x_2 + x_3}{\sqrt{3}} \right)^2 + 2 \cdot y_3^2$$

Note that the symbols λ_1 and y_3 in this presentation of $q(\boldsymbol{x})$ are currently unknown.

(6 pts) (*a*) $\|\boldsymbol{a}_1\|^2 = \|\boldsymbol{a}_3\|^2 =$ _______ and $\langle \boldsymbol{a}_1, \boldsymbol{a}_2 \rangle =$ _______

(9 pts) (*b*) Find the value of λ_1. Clearly explain your reasoning to receive credit. Fill your answer in the blank below for clarity.

$\lambda_1 =$ _______

(6 pts) (*c*) There are only two valid formulas for y_3 in terms of x_1, x_2, and x_3. Find one of these formulas. Clearly explain your reasoning to receive credit. Fill in the blank below for clarity.

$y_3 =$ ______________________________